arXiv:1508.03034v1 [math.RA] 12 Aug 2015

Automorphic Equivalence in the Varieties of Representations of Lie algebras.

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August 13, 2015

Abstract

In this paper we consider the very wide class of varieties of representations of Lie algebras over the field k, which has characteristic 0. We study the relation between the geometric equivalence and automorphic equivalence of the representations of these varieties.

If we denote by Θ one of these varieties, then Θ^0 is a category of the finite generated free representations of the variety Θ. In this paper, we calculate for the considered varieties the quotient group $\mathfrak{A}/\mathfrak{Y}$, where $\mathfrak{A}$ is a group of all the automorphisms of the category Θ^0 and $\mathfrak{Y}$ is a subgroup of all the inner automorphisms of this category. The quotient group $\mathfrak{A}/\mathfrak{Y}$ measures the difference between the geometric equivalence and automorphic equivalence of representations of the variety Θ.

In [8], the situation when Θ is the variety of all the representations of Lie algebras over the field k, which has characteristic 0, was considered. The problem was resolved by the reduction to some variety of the one-sorted algebraic structures, so, the considerations were somewhat long and sophisticated. The many-sorted approach to the method of verbal operations for computation of the quotient group $\mathfrak{A}/\mathfrak{Y}$ was elaborated on in [12]. By this approach, the result of [8] was proven again in [12] in a simpler manner. The method of [12] allows us to achieve in this paper a result for the wide class of subvarieties of the variety of all the representations of Lie algebras.

In other classes of algebraic structures, we have a completely different situation. In the variety of all the groups, a similar result was achieved only for the subvariety of all the Abelian groups [7] and for the subvarieties of all the nilpotent groups of the class $\leq d$, where $d \in \mathbb{N}$, $d \geq 2$ [11]. In the theory of the representations of groups a similar result was achieved only for the variety of all the representations of groups in [7] and proved again in [12].

In Section 5, we present one example of the subvariety Θ of the variety of all the representations of the Lie algebras over the field k, and two representations from the variety Θ which are automorphically equivalent but not geometrically equivalent.

1 Introduction

A compilation of all the definitions of the basic notions of the universal algebraic geometry can be found, for example, in [4], [5] and [6]. Also, there are the fundamental articles [2] and [3]. The natural question of the universal algebraic geometry is: "When do two universal algebras H_1 and H_2 from the some variety Θ have the same algebraic geometry"? But, first of all, what does it mean that "two algebras have same algebraic geometry"? The two notions of geometric equivalence and automorphic equivalence can give a answer to this question. This article focuses on the relationship between these notions in a very wide class of the varieties of the representations of the Lie algebras over the field k, which has characteristic 0.

We consider the representations of the Lie algebras as two-sorted universal algebras: the first sort is a sort of elements of Lie algebras, and the second sort is a sort of vectors of linear spaces. Therefore, we will consider all basic notions of the universal algebraic geometry in the many-sorted version as in [12]. We suppose that there is a finite set of names of sorts Γ. In our case $\Gamma = \{1, 2\}$. Many-sorted algebra, first of all, is a set H with the "sorting": mapping $\eta_H : H \to \Gamma$. The set of elements of the sort i, where $i \in \Gamma$, of the algebra H will be the set $\eta_H^{-1}(i)$. We denote $\eta_H^{-1}(i) = H^{(i)}$. If $h \in H^{(i)}$, then many times we will denote $h = h^{(i)}$, with a view to emphasizing that h is an element of the sort i.

We denote by Ω the signature (set of operations) of our algebras. Every operation $\omega \in \Omega$ has a type $\tau_\omega = (i_1, \ldots, i_n; j)$, where $n \in \mathbb{N}$, $i_1, \ldots, i_n, j \in \Gamma$. Operation $\omega \in \Omega$ of the type $(i_1, \ldots, i_n; j)$ is a partially defined mapping $\omega : H^n \to H$. This mapping is defined only for tuples $(h_1, \ldots, h_n) \in H^n$ such that $h_k \in H^{(i_k)}$, $1 \leq k \leq n$. The images of these tuples are elements of the sort j: $\omega(h_1, \ldots, h_n) \in H^{(j)}$.

In our case the signature Ω of the representations of the Lie algebras has this form:

$$\Omega = \left\{ 0^{(1)}, -^{(1)}, \lambda^{(1)}(\lambda \in k), +^{(1)}, [,], 0^{(2)}, -^{(2)}, \lambda^{(2)}(\lambda \in k), +^{(2)}, \circ \right\}. \quad (1.1)$$

$0^{(2)}$ is the 0-ary operation of taking the zero vector in the linear space, $\tau_{0^{(2)}} = (2)$. $-^{(2)}$ is the unary operation of taking the negative vector in the linear space, $\tau_{-^{(2)}} = (1; 2)$. $+^{(2)}$ is the operation of addition of the vectors of the linear space, $\tau_{+^{(2)}} = (2, 2; 2)$. For every $\lambda \in k$ we have the unary operation of multiplication of vectors from the linear space by the scalar λ. We denote this operation by λ and $\tau_\lambda = (2; 2)$. $0^{(1)}$, $-^{(1)}$, $\lambda^{(1)}(\lambda \in k)$, $+^{(1)}$ are the similar operations in the Lie algebra. $[,]$ is the Lie brackets; this operation has type $\tau_{[,]} = (1, 1; 1)$. $\circ$ is

an operation of the action of elements of the Lie algebra on vectors from the linear space, $\tau_\circ = (1, 2; 2)$.

In universal algebraic geometry we consider some variety Θ of universal algebras of the signature Ω. We denote by $X_0 = \bigcup_{i\in\Gamma} X_0^{(i)}$ a set of symbols, such that $X_0^{(i)}$ is an infinite countable set for every $i \in \Gamma$ and $X_0^{(i)} \cap X_0^{(j)} = \varnothing$ when $i \neq j$. By $\mathfrak{F}(X_0)$ we denote the set of all finite subsets of X_0. We will consider the category Θ^0, whose objects are all free algebras $F(X)$ of the variety Θ generated by finite subsets $X \in \mathfrak{F}(X_0)$, such that $(F(X))^{(i)} \supseteq X \cap X_0^{(i)}$. Morphisms of the category Θ^0 are homomorphisms of these algebras. We will occasionally denote $F(X) = F(x_1, x_2, \ldots, x_n)$ if $X = \{x_1, x_2, \ldots, x_n\}$ and even $F(X) = F(x)$ if X has only one element.

We consider a "system of equations" $T \subseteq \bigcup_{i\in\Gamma} \left((F)^{(i)}\right)^2$, where $F \in \mathrm{Ob}\Theta^0$ (see [12, Section 4]), and we "resolve" these equations in arbitrary algebra $H \in \Theta$. The set $\mathrm{Hom}(F, H)$ serves as an "affine space over the algebra H": the solution of the system T is a homomorphism $\mu \in \mathrm{Hom}(F, H)$ such that $\mu(t_1) = \mu(t_2)$ holds for every $(t_1, t_2) \in T$ or $T \subseteq \ker\mu$. $T'_H = \{\mu \in \mathrm{Hom}(F, H) \mid T \subseteq \ker\mu\}$ will be the set of all the solutions of the system T. For every set of "points" $R \subseteq \mathrm{Hom}(F, H)$ we consider a congruence of equations defined in this way: $R'_H = \bigcap_{\mu\in R} \ker\mu$. This is a maximal system of equations which has the set of solutions R. For every set of equations T we consider its algebraic closure $T''_H = \bigcap_{\mu\in T'_H} \ker\mu$ with respect to the algebra H. In the many-sorted case it is possible that $\mathrm{Hom}(F, H) = \varnothing$, and in this situation $T''_H = \bigcup_{i\in\Gamma} \left((F)^{(i)}\right)^2$ holds for every $T \subseteq \bigcup_{i\in\Gamma} \left((F)^{(i)}\right)^2$. A set $T \subseteq \bigcup_{i\in\Gamma} \left((F)^{(i)}\right)^2$ is called H-closed if $T = T''_H$. An H-closed set is always a congruence. We denote the family of all H-closed congruences in F by $Cl_H(F)$.

Definition 1.1 *Algebras $H_1, H_2 \in \Theta$ are **geometrically equivalent** if and only if for every $F \in \mathrm{Ob}\Theta^0$ and every $T \subseteq \bigcup_{i\in\Gamma} \left((F)^{(i)}\right)^2$ the equality $T''_{H_1} = T''_{H_2}$ is fulfilled.*

By this definition, algebras $H_1, H_2 \in \Theta$ are geometrically equivalent if and only if the families $Cl_{H_1}(F)$ and $Cl_{H_2}(F)$ coincide for every $F \in \mathrm{Ob}\Theta^0$.

Definition 1.2 *[6]We say that algebras $H_1, H_2 \in \Theta$ are **automorphically equivalent** if there exist an automorphism $\Phi : \Theta^0 \to \Theta^0$ and the bijections*

$$\alpha(\Phi)_F : Cl_{H_1}(F) \to Cl_{H_2}(\Phi(F))$$

for every $F \in \mathrm{Ob}\Theta^0$, coordinated in the following sense: if $F_1, F_2 \in \mathrm{Ob}\Theta^0$,

$\mu_1, \mu_2 \in \mathrm{Hom}\,(F_1, F_2)$, $T \in Cl_{H_1}(F_2)$ *then*

$$\tau\mu_1 = \tau\mu_2,$$

if and only if

$$\widetilde{\tau}\Phi\,(\mu_1) = \widetilde{\tau}\Phi\,(\mu_2)\,,$$

where $\tau : F_2 \to F_2/T$, $\widetilde{\tau} : \Phi\,(F_2) \to \Phi\,(F_2)\,/\alpha(\Phi)_{F_2}\,(T)$ *are the natural epimorphisms.*

The definition of the automorphic equivalence in the language the category of coordinate algebras was considered in [6] and [12]. Intuitively we can say that algebras $H_1, H_2 \in \Theta$ are automorphically equivalent if and only if the families $Cl_{H_1}(F)$ and $Cl_{H_2}(\Phi\,(F))$ coincide up to a change of coordinates. This change is defined by the automorphism Φ.

Definition 1.3 *An automorphism Υ of an arbitrary category $\mathfrak{K}$ is* ***inner****, if it is isomorphic as a functor to the identity automorphism of the category $\mathfrak{K}$.*

It means that for every $F \in \mathrm{Ob}\mathfrak{K}$ there exists an isomorphism $\sigma_F^{\Upsilon} : F \to \Upsilon\,(F)$ such that for every $\mu \in \mathrm{Mor}_{\mathfrak{K}}\,(F_1, F_2)$

$$\Upsilon\,(\mu) = \sigma_{F_2}^{\Upsilon}\mu\left(\sigma_{F_1}^{\Upsilon}\right)^{-1}$$

holds. It is clear that the set $\mathfrak{Y}$ of all inner automorphisms of an arbitrary category $\mathfrak{K}$ is a normal subgroup of the group $\mathfrak{A}$ of all automorphisms of this category.

By [6, Proposition 9] and [12, Theorem 4.2] (many-sorted case), if an inner automorphism Υ provides the automorphic equivalence of the algebras H_1 and H_2, where $H_1, H_2 \in \Theta$, then H_1 and H_2 are geometrically equivalent. Therefore the quotient group $\mathfrak{A}/\mathfrak{Y}$ measures the possible difference between the geometric equivalence and automorphic equivalence of algebras from the variety Θ.

From now on, the word "representation" means a representation of the Lie algebra over the field k, which has characteristic 0.

We will use the method elaborated on in [7] to the one-sorted algebras and in [12] to the many-sorted algebras for the calculation of the quotient group $\mathfrak{A}/\mathfrak{Y}$ for the wide class of varieties of representations. To use this method, we study in Section 2 the structure of the free representations in the varieties of representations. Then we will study in Section 3 some properties of the category Θ^0, where Θ is a variety of representations.

2 Homogenization of the identities in the representations of the Lie algebras

In this section we want to clarify and generalize a few the considerations which can be seen in the beginning of [9].

We consider an absolutely free representation $F(X)$ generated by the set $X = X^{(1)} \cup X^{(2)}$, such that $(F(X))^{(i)} \supseteq X^{(i)}$, $i = 1, 2$. $(F(X))^{(1)} = L\left(X^{(1)}\right)$ is a free Lie algebra generated by the set $X^{(1)}$. $(F(X))^{(2)} = A\left(X^{(1)}\right) X^{(2)}$, where $A\left(X^{(1)}\right)$ is a free associative algebra with unit generated by the set $X^{(1)}$ and

$$A\left(X^{(1)}\right) X^{(2)} = \bigoplus_{x^{(2)} \in X^{(2)}} A\left(X^{(1)}\right) x^{(2)} =$$

$$\mathrm{Sp}_k \left\{ x_{i_n}^{(1)} \dots x_{i_1}^{(1)} x^{(2)} \mid x_{i_j}^{(1)} \in X^{(1)}, x^{(2)} \in X^{(2)} \right\}$$

is a free left $A\left(X^{(1)}\right)$-module generated by the set $X^{(2)}$. For every $l \in (F(X))^{(1)} = L\left(X^{(1)}\right)$ and every $v \in (F(X))^{(2)}$ we understand $l \circ v$ as $\iota(l) v$, where $\iota : L\left(X^{(1)}\right) \to A\left(X^{(1)}\right)$ is an embedding, which exists by the Poincaré - Birkhoff - Witt theorem.

Now we consider an arbitrary subvariety Θ of the variety of all the representations. Free representation $F_\Theta(X)$ of Θ generated by the set $X = X^{(1)} \cup X^{(2)}$ is the representation $F(X) / \mathrm{Id}_\Theta(X)$, where $\mathrm{Id}_\Theta(X)$ is a congruence of all the identities of the variety Θ which contain variables from the set X. Actually, the free generators of the representation $F_\Theta(X)$ have a form $\nu(x)$, where $\nu : F(X) \to F(X) / \mathrm{Id}_\Theta(X)$ is a natural epimorphism, X is a set of free generators of the representation $F(X)$, $x \in X$. But we will use the same symbols from the free generators of the representations $F(X)$ and $F_\Theta(X)$. $(F_\Theta(X))^{(1)} = L\left(X^{(1)}\right) / I_\Theta\left(X^{(1)}\right)$, $(F_\Theta(X))^{(2)} = A\left(X^{(1)}\right) X^{(2)} / V_\Theta(X)$, where $I_\Theta\left(X^{(1)}\right)$ is an ideal of the $L\left(X^{(1)}\right)$, $V_\Theta(X)$ is a $A\left(X^{(1)}\right)$-left submodule of the $A\left(X^{(1)}\right) X^{(2)}$ and for every $l \in I_\Theta\left(X^{(1)}\right)$ and for every $v \in (F_\Theta(X))^{(2)}$ the $l \circ v \in V_\Theta(X)$ holds. The ideal $I_\Theta\left(X^{(1)}\right)$ and the submodule $V_\Theta(X)$ are fully invariant. The ideal $I_\Theta\left(X^{(1)}\right)$ is polyhomogeneous by the Theorem of the homogenization of the identities of linear algebras (see for example [1, Theorem 4.2.2]). So the Lie algebra $L_\Theta\left(X^{(1)}\right) = L\left(X^{(1)}\right) / I_\Theta\left(X^{(1)}\right)$ is a graded algebra.

$(F(X))^{(2)} = \bigoplus\limits_{i \in I} U_i$, where $X^{(2)} = \left\{ x_i^{(2)} \mid i \in I \right\}$, $U_i = A\left(X^{(1)}\right) x_i^{(2)}$ is a free left $A\left(X^{(1)}\right)$-cyclic module. Every element $u \in (F(X))^{(2)}$ has unique decomposition $u = \sum\limits_{i \in I_u} u_i$, where $I_u \subseteq I$, $|I_u| < \infty$, $u_i = f_i x_i^{(2)} \in U_i$, $f_i \in A\left(X^{(1)}\right)$. We shell call the elements u_i the **cyclic components** of the element u.

Proposition 2.1 *If $v \in V_\Theta(X)$, then all cyclic components of the v are also elements of $V_\Theta(X)$.*

Proof. We will consider a decomposition of v to the cyclic components: $v = \sum\limits_{i \in I_v} v_i = \sum\limits_{i \in I_v} f_i x_i^{(2)}$. For every $i \in I_v$, there exists $\chi_i \in \mathrm{End}(F(X))$ such that $(\chi_i)_{|X^{(1)}} = id_{X^{(1)}}$, $\chi_i\left(x_i^{(2)}\right) = x_i^{(2)}$, $\chi_i\left(x_j^{(2)}\right) = 0^{(2)}$ for every $j \in I \setminus \{i\}$. $\chi_i(v) = v_i \in V_\Theta(X)$. ∎

Therefore $V_\Theta(X) = \bigoplus_{i\in I} (U_i \cap V_\Theta(X))$ and

$$(F_\Theta(X))^{(2)} = (F(X))^{(2)} / V_\Theta(X) \cong \bigoplus_{i\in I} (U_i / (U_i \cap V_\Theta(X))).$$

$U_i \cap V_\Theta(X) = N_i = S_i x_i^{(2)}$, where S_i is a left-side ideal of $A\left(X^{(1)}\right)$.

Proposition 2.2 *S_i is a two-sided polyhomogeneous ideal of $A\left(X^{(1)}\right)$.*

Proof. If $s \in S_i$, then $sx_i^{(2)} \in V_\Theta(X)$. We take $f \in A\left(X^{(1)}\right)$. There exists $\chi_f \in \mathrm{End}(F(X))$ such that $\left(\chi_f\right)_{|X^{(1)}} = id_{X^{(1)}}$, $\chi_f\left(x_i^{(2)}\right) = fx_i^{(2)}$, $\chi_f\left(x_j^{(2)}\right) = x_j^{(2)}$ for every $j \in I \setminus \{i\}$. $\chi_f\left(sx_i^{(2)}\right) = sfx_i^{(2)} \in V_\Theta(X)$, so $sf \in S_i$.

The proof of the fact that S_i is a polyhomogeneous ideal is a very similar to the proof of the theorem of the homogenization of the identities of linear algebras. ∎

Therefore $U_i / (U_i \cap V_\Theta(X)) = A\left(X^{(1)}\right) x_i^{(2)} / S_i x_i^{(2)} \cong \left(A\left(X^{(1)}\right)/S_i\right) x_i^{(2)}$, where $A\left(X^{(1)}\right)/S_i = A_i$ is a graded algebra.

Proposition 2.3 *There exists a two-sided polyhomogeneous ideal $S_\Theta\left(X^{(1)}\right) \leq A\left(X^{(1)}\right)$, such that $V_\Theta(X) = S_\Theta\left(X^{(1)}\right) X^{(2)} = \bigoplus_{i\in I} S_\Theta\left(X^{(1)}\right) x_i^{(2)}$.*

Proof. We only need to prove that $S_i = S_j$ for every $i, j \in I$. We consider $s \in S_i$. $sx_i^{(2)} \in U_i \cap V_\Theta(X)$. There exists $\chi_j \in \mathrm{End}(F(X))$ such that $\left(\chi_j\right)_{|X^{(1)}} = id_{X^{(1)}}$, $\chi_j\left(x_i^{(2)}\right) = x_j^{(2)}$, $\chi_j\left(x_k^{(2)}\right) = 0^{(2)}$ for every $k \in I \setminus \{i\}$. $\chi_j\left(sx_i^{(2)}\right) = sx_j^{(2)} \in U_j \cap V_\Theta(X)$. Therefore $s \in S_j$. ∎

Therefore we prove the following:

Theorem 2.1 $(F_\Theta(X))^{(1)} = L_\Theta\left(X^{(1)}\right) = L\left(X^{(1)}\right)/I_\Theta\left(X^{(1)}\right)$, $(F_\Theta(X))^{(2)} = \bigoplus_{x\in X^{(2)}} \left(A_\Theta\left(X^{(1)}\right) x\right)$, *where* $A_\Theta\left(X^{(1)}\right) = A\left(X^{(1)}\right)/S_\Theta\left(X^{(1)}\right)$, $L\left(X^{(1)}\right)$ *is a free Lie algebra, generated by the set* $X^{(1)}$, $I_\Theta\left(X^{(1)}\right)$ *is a polyhomogeneous ideal of this algebra,* $A\left(X^{(1)}\right)$ *is a free associative algebra with unit, generated by the set* $X^{(1)}$, $S_\Theta\left(X^{(1)}\right)$ *is a polyhomogeneous two-sided ideal of this algebra,* $A_\Theta\left(X^{(1)}\right) = A\left(X^{(1)}\right)/S_\Theta\left(X^{(1)}\right)$ *is a graded algebra. Also for every* $a \in A\left(X^{(1)}\right)$ *and every* $l \in I_\Theta\left(X^{(1)}\right)$ *is fulfilled* $al, la \in S_\Theta\left(X^{(1)}\right)$ *(*$I_\Theta\left(X^{(1)}\right) \subseteq \left(S_\Theta\left(X^{(1)}\right) : A\left(X^{(1)}\right)\right)$*).*

3 Category of the finitely generated free representations

We consider the category Θ^0, where Θ is an arbitrary subvariety of the variety of all the representations.

Definition 3.1 *We say that the variety Θ has an* ***IBN propriety*** *if for every $F_{\Theta}(X), F_{\Theta}(Y) \in \mathrm{Ob}\Theta^{0}$ the $F_{\Theta}(X) \cong F_{\Theta}(Y)$ holds if and only if $\left|X^{(i)}\right| = \left|Y^{(i)}\right|$, $i = 1, 2$.*

We consider the nontrivial variety Θ. It means that in the variety Θ the identity $x^{(2)} = 0$ is not fulfilled.

Proposition 3.1 *Every nontrivial variety Θ has IBN propriety.*

Proof. We consider $F_{\Theta}(X) = F_{\Theta} \in \mathrm{Ob}\Theta^{0}$. We will denote $A\left(X^{(1)}\right) = A$, $F_{\Theta}^{(1)} = L_{\Theta}\left(X^{(1)}\right) = L_{\Theta}$, $A_{\Theta}\left(X^{(1)}\right) = A_{\Theta}$, $S_{\Theta}\left(X^{(1)}\right) = S_{\Theta}$. We denote by J the two-sided ideal of A generated by set $X^{(1)}$: $J = \left\langle X^{(1)} \right\rangle_{ideal A}$.

By [13, Section 3] we have

$$\left|X^{(1)}\right| = \dim_{k}\left(L_{\Theta}/\left[L_{\Theta}, L_{\Theta}\right]\right).$$

We will use the description of $F_{\Theta}(X)$ given in Theorem 2.1. $\left(F_{\Theta}^{(1)}\right) \circ \left(F_{\Theta}^{(2)}\right) = L_{\Theta} \circ \left(\bigoplus_{x \in X^{(2)}} A_{\Theta} x\right) = \bigoplus_{x \in X^{(2)}} \left(L_{\Theta} \cdot A_{\Theta}\right) x$. $L_{\Theta} \cdot A_{\Theta}$ is a two-sided ideal of A_{Θ}, because $X^{(1)} \subset L_{\Theta}$. Θ is the nontrivial variety, so $J \supseteq S_{\Theta}$ and $L_{\Theta} \cdot A_{\Theta} = J/S_{\Theta}$. $A_{\Theta}/\left(L_{\Theta} \cdot A_{\Theta}\right) = \left(A/S_{\Theta}\right)/\left(J/S_{\Theta}\right) \cong A/J \cong k$.

$$F_{\Theta}^{(2)}/\left(F_{\Theta}^{(1)} \circ F_{\Theta}^{(2)}\right) = \left(\bigoplus_{x \in X^{(2)}} A_{\Theta} x\right) / \left(\bigoplus_{x \in X^{(2)}} \left(L_{\Theta} \cdot A_{\Theta}\right) x\right) \cong$$

$$\bigoplus_{x \in X^{(2)}} \left(A_{\Theta}/\left(L_{\Theta} \cdot A_{\Theta}\right)\right) x,$$

hence

$$\dim_{k}\left(F_{\Theta}^{(2)}/\left(F_{\Theta}^{(1)} \circ F_{\Theta}^{(2)}\right)\right) = \left|X^{(2)}\right|.$$

■

We say that the variety Θ is an action-type variety, if $I_{\Theta}\left(X^{(1)}\right) = \{0\}$ for every $X^{(1)} \subset X_{0}^{(1)}$, such that $\left|X^{(1)}\right| < \infty$.

Proposition 3.2 *If Θ is an action-type nontrivial variety, then $\Phi\left(F_{\Theta}\left(x^{(1)}\right)\right) = F_{\Theta}\left(x^{(1)}\right)$ and $\Phi\left(F_{\Theta}\left(x^{(2)}\right)\right) = F_{\Theta}\left(x^{(2)}\right)$ hold for every $\Phi \in \mathrm{Aut}\Theta^{0}$.*

Proof. Θ has IBN propriety, so, by [8, Proposition 5.2] and [12, Section 5], we have two possibilities for every $\Phi \in \mathrm{Aut}\Theta^{0}$: or $\Phi\left(F_{\Theta}\left(x^{(1)}\right)\right) = F_{\Theta}\left(x^{(1)}\right)$ and $\Phi\left(F_{\Theta}\left(x^{(2)}\right)\right) = F_{\Theta}\left(x^{(2)}\right)$, or $\Phi\left(F_{\Theta}\left(x^{(1)}\right)\right) = F_{\Theta}\left(x^{(2)}\right)$ and $\Phi\left(F_{\Theta}\left(x^{(2)}\right)\right) = F_{\Theta}\left(x^{(1)}\right)$. $\left(F_{\Theta}\left(x_{1}^{(1)}, \ldots, x_{n}^{(1)}\right)\right)^{(1)} = L\left(x_{1}^{(1)}, \ldots, x_{n}^{(1)}\right)$, $\left(F_{\Theta}\left(x_{1}^{(1)}, \ldots, x_{n}^{(1)}\right)\right)^{(2)} = \{0\}$; $\left(F_{\Theta}\left(x_{1}^{(2)}, \ldots, x_{n}^{(2)}\right)\right)^{(1)} = \{0\}$, $\left(F_{\Theta}\left(x_{1}^{(2)}, \ldots, x_{n}^{(2)}\right)\right)^{(2)} = \mathrm{Sp}_{k}\left(x_{1}^{(2)}, \ldots, x_{n}^{(2)}\right)$. Now we can use the argument of [12, Proposition 5.9] and conclude that there does not exist $\Phi \in \mathrm{Aut}\Theta^{0}$ such that $\Phi\left(F_{\Theta}\left(x^{(1)}\right)\right) = F_{\Theta}\left(x^{(2)}\right)$ and $\Phi\left(F_{\Theta}\left(x^{(2)}\right)\right) = F_{\Theta}\left(x^{(1)}\right)$. ■

4 Method of the verbal operations

In the beginning of this section we will explain the method of [7] and [12] in the case of arbitrary variety Θ of universal algebras of the signature Ω.

In [7] the notion of the strongly stable automorphism of the category Θ^0 was defined. In the case of the variety of many-sorted algebras ($|\Gamma| > 1$) we have the following:

Definition 4.1 *[12]An automorphism Φ of the category Θ^0 is called **strongly stable** if it satisfies the conditions:*

1 Φ preserves all objects of Θ^0,

2 there exists a system of bijections $S = \left\{s_F : F \rightarrow F \mid F \in \mathrm{Ob}\Theta^0\right\}$ such that all these bijections conform with the sorting:

$$\eta_F = \eta_F s_F$$

3 Φ acts on the morphisms $\mu \in \mathrm{Mor}_{\Theta^0}(F_1, F_2)$ of Θ^0 thusly:

$$\Phi(\mu) = s_{F_2} \mu s_{F_1}^{-1},$$

4 $s_F \mid_X = id_X$, for every $F(X) \in \mathrm{Ob}\Theta^0$.

It is clear that the set $\mathfrak{S}$ of all strongly stable automorphisms of the category Θ^0 is a subgroup of the group $\mathfrak{A}$ of all automorphisms of this category. By [12, Theorem 2.3], $\mathfrak{A} = \mathfrak{Y}\mathfrak{S}$ holds if in the category Θ^0 the

Condition 4.1 *$\Phi\left(F\left(x^{(i)}\right)\right) \cong F\left(x^{(i)}\right)$ for every automorphism Φ of the category Θ^0, every sort $i \in \Gamma$ and every $x^{(i)} \in X_0^{(i)} \subset X_0$*

holds. In this case we have that $\mathfrak{A}/\mathfrak{Y} \cong \mathfrak{S}/\mathfrak{S} \cap \mathfrak{Y}$. So we must compute the groups $\mathfrak{S}$ and $\mathfrak{S} \cap \mathfrak{Y}$.

The group $\mathfrak{S}$ we can compute by the method of verbal operations. For every word $w = w(x_1, \ldots, x_n) \in F(x_1, \ldots, x_n) = F \in \mathrm{Ob}\Theta^0$ and every algebra $H \in \Theta$ we can define an operation w_H^* in H: if $h_1, \ldots, h_n \in H$ such that $\eta_H(h_i) = \eta_F(x_i)$, where $1 \leq i \leq n$, then $w_H^*(h_1, \ldots, h_n) = \alpha(w)$, where $\alpha : F \rightarrow H$ is a homomorphism such that $\alpha(x_i) = h_i$. If $\eta_F\{x_1, \ldots, x_n\} \nsubseteq \Gamma_H$ then the operation w_H^* is defined on the empty subset of H^n. The operation w_H^* is called the verbal operation defined by the word w. This operation we consider as the operation of the type $(\eta_F(x_1), \ldots, \eta_F(x_n); \eta_F(w))$ even if not all of the free generators $x_1, \ldots, x_n$ really enter into the word w.

If we have a system of words $W = \{w_i \mid i \in I\}$ such that $w_i \in F_i \in \mathrm{Ob}\Theta^0$, then for every $H \in \Theta$ we denote by H_W^* the universal algebra which coincides with H as a set with "sorting", but has only verbal operations defined by the words from W.

For the operation $\omega \in \Omega$ which has a type $\tau_\omega = (i_1, \ldots, i_n; j)$, we take $F_\omega = F(X_\omega) \in \mathrm{Ob}\Theta^0$ such that $X_\omega = \left\{x^{(i_1)}, \ldots, x^{(i_n)}\right\}$, $\eta_{A_\omega}\left(x^{(i_k)}\right) = i_k$,

$1 \leq k \leq n$. By the method of the verbal operations (see [7], [10] and [12]) there is a bijection between the set of the strongly stable automorphisms of the category Θ^0 and the set of the systems of words W which fulfill the

Condition 4.2 *1. $W = \{w_\omega \in F_\omega \mid \omega \in \Omega\}$,*

2. for every $F(X) \in \mathrm{Ob}\Theta^0$ there exists a bijection $s_F : F \to F$ such that $(s_F)_{|X} = id_X$ and $s_F : F \to F_W^*$ is an isomorphism.

We can compute the group $\mathfrak{S} \cap \mathfrak{Y}$ by this

Criterion 4.1 *[12, Proposition 3.7]The strongly stable automorphism Φ which corresponds to the system of words $W^\Phi = W$ is inner if and only if there is a system of isomorphisms $\left\{\tau_F : F \to F_W^* \mid F \in \mathrm{Ob}\Theta^0\right\}$ such that for every $F_1, F_2 \in \mathrm{Ob}\Theta^0$ and every $\mu \in \mathrm{Mor}_{\Theta^0}(F_1, F_2)$ the*

$$\tau_{F_2}\mu = \mu\tau_{F_1}$$

holds.

Now we came back to the varieties of the representations. By Proposition 3.2 we have that in an action-type nontrivial variety of the representations, Condition 4.1 is fulfilled. The signature Ω of the representations was described in (1.1). For every $\omega \in \Omega$ we must find all the possible forms of the words w_ω such that the system $W = \{w_\omega \mid \omega \in \Omega\}$ fulfills Condition 4.2.

We say that the variety of the representations is degenerated if the identity $\left[x_1^{(1)}, x_2^{(1)}\right] \circ x^{(2)} = 0$ is fulfilled in this variety. It is clear that the nondegenerated variety is nontrivial.

Proposition 4.1 *If Θ is an action-type and nondegenerated variety of representations, then the system $W = \{w_\omega \mid \omega \in \Omega\}$, which fulfills Condition 4.2 has a form*

$$w_{0^{(1)}} = 0^{(1)}, w_{-^{(1)}} = -x^{(1)}, w_{\lambda^{(1)}} = \varphi(\lambda)\, x^{(1)},$$

$$w_{+^{(1)}} = x_1^{(1)} + x_2^{(1)}, w_{[,]} = a\left[x_1^{(1)}, x_2^{(1)}\right], \tag{4.1}$$

$$w_{0^{(2)}} = 0^{(2)}, w_{-^{(2)}} = -x^{(2)}, w_{\lambda^{(2)}} = \varphi(\lambda)\, x^{(2)},$$

$$w_{+^{(2)}} = x_1^{(2)} + x_2^{(2)}, w_\circ = a\left(x^{(1)} \circ x^{(2)}\right),$$

where $\varphi \in \mathrm{Aut}k$, $a \in k^*$.

Proof. By computations in $F_\Theta(\varnothing)$, $F_\Theta\left(x^{(1)}\right)$ and $F_\Theta\left(x_1^{(1)}, x_2^{(1)}\right)$ we can conclude that $w_{0^{(1)}} = 0^{(1)}$, $w_{-^{(1)}} = -x^{(1)}$, $w_{\lambda^{(1)}} = \varphi(\lambda)\, x^{(1)}$, $w_{+^{(1)}} = x_1^{(1)} + x_2^{(1)}$, $w_{[,]} = a\left[x_1^{(1)}, x_2^{(1)}\right]$, where $\varphi \in \mathrm{Aut}k$, $a \in k^*$. These computations can be seen in [7] and [13, Section 4]. By the computations in $F_\Theta(\varnothing)$, $F_\Theta\left(x^{(2)}\right)$

and $F_\Theta\left(x_1^{(2)},x_2^{(2)}\right)$ we can conclude that $w_{0^{(2)}}=0^{(2)}$, $w_{-^{(2)}}=-x^{(2)}$, $w_{\lambda^{(2)}}=\psi\left(\lambda\right)x^{(2)}$, $w_{+^{(2)}}=x_1^{(2)}+x_2^{(2)}$, where $\psi\in\mathrm{Aut}k$. These computations are very simple. $w_\circ$ must be calculated in $F_\Theta\left(x^{(1)},x^{(2)}\right)$. Θ is a nondegenerated variety, so the identity $x^{(1)}\circ x^{(2)}=0$ is not fulfilled in this variety. Therefore, by Theorem 2.1 $A_\Theta\left(x^{(1)}\right)=A\left(x^{(1)}\right)/S_\Theta\left(x^{(1)}\right)=k\left[x^{(1)}\right]/\left(\left(x^{(1)}\right)^d\right)$, where $d>1$, and $w_\circ\left(x_1^{(2)},x_2^{(2)}\right)=f\left(x^{(1)}\right)\circ x^{(2)}$, where $f\left(x^{(1)}\right)\in k\left[x^{(1)}\right]$, $\deg f\left(x^{(1)}\right)<d$.

We use the method of the [12, Theorem 5.4]. In $F_\Theta\left(x^{(1)},x^{(2)}\right)$ the equality $\lambda\left(x^{(1)}\circ x^{(2)}\right)=\left(\lambda x^{(1)}\right)\circ x^{(2)}$ holds for every $\lambda\in k$. If there exists a bijection $s_F:F\to F$, where $F=F_\Theta\left(x^{(1)},x^{(2)}\right)$, such that $\left(s_F\right)_{|X}=id_X$, where $X=\left\{x^{(1)},x^{(2)}\right\}$, and $s_F:F\to F_W^*$ is an isomorphism, then $s_F\left(\lambda\left(x^{(1)}\circ x^{(2)}\right)\right)=w_{\lambda^{(2)}}\left(w_\circ\left(x^{(1)},x^{(2)}\right)\right)$ and $s_F\left(\left(\lambda x^{(1)}\right)\circ x^{(2)}\right)=w_\circ\left(w_{\lambda^{(1)}}\left(x^{(1)}\right),x^{(2)}\right)$. Now we can conclude that $w_\circ\left(x_1^{(2)},x_2^{(2)}\right)=bx^{(1)}\circ x^{(2)}$, where $b\in k\setminus\{0\}$.

We consider the nondegenerated variety. Thus, by the same method, we conclude that $a=b$ from the equality $\left[x_1^{(1)},x_2^{(1)}\right]\circ x^{(2)}=x_1^{(1)}\circ\left(x_2^{(1)}\circ x^{(2)}\right)-x_2^{(1)}\circ\left(x_1^{(1)}\circ x^{(2)}\right)$ in $F_\Theta\left(x_1^{(1)},x_2^{(1)},x^{(2)}\right)$.

Also we conclude $\varphi=\psi$ from the equality $\left(\lambda x^{(1)}\right)\circ x^{(2)}=x^{(1)}\circ\left(\lambda x^{(2)}\right)$ which holds in $F_\Theta\left(x^{(1)},x^{(2)}\right)$ for every $\lambda\in k$. ■

Theorem 4.1 *If Θ is an action-type and nondegenerated variety of representations and this variety is defined by identities with coefficients from $\mathbb{Z}$, then $\mathfrak{A}/\mathfrak{Y}\cong\mathrm{Aut}k$.*

Proof. Now we will prove that Condition 4.2 is fulfilled for the systems of words W which have a form (4.1). We will consider an absolutely free representation $F\left(X\right)=F$ generated by the set $X=X^{(1)}\cup X^{(2)}$, such that $F^{(i)}\supset X^{(i)}$, $i=1,2$, where $X\in\mathfrak{F}\left(X_0\right)$. By [12, Theorem 5.4], there exists an isomorphism $\widetilde{s}_F:F\to F_W^*$, such that $\left(\widetilde{s}_F\right)_{|X}=id_X$. There exists the natural epimorphism $\nu:F\left(X\right)\to F_\Theta\left(X\right)=F\left(X\right)/\mathrm{Id}_\Theta\left(X\right)$, where $F_\Theta\left(X\right)=F_\Theta$ is a free representation of the variety Θ generated by the set X. In truth, $F_\Theta\left(X\right)$ is generated by the set $\nu\left(X\right)$ but we will use our short notation. The operations defined in the representations F_W^* and $\left(F_\Theta\right)_W^*$ are the verbal operations, so $\nu:F_W^*\to\left(F_\Theta\right)_W^*$ is also an epimorphism. By Theorem 2.1,

$$\left(F_\Theta\left(X\right)\right)^{(1)}=L\left(X^{(1)}\right),\left(F_\Theta\left(X\right)\right)^{(2)}=\bigoplus_{x\in X^{(2)}}\left(\left(A\left(X^{(1)}\right)/S_\Theta\left(X^{(1)}\right)\right)x\right),$$

where $S_\Theta\left(X^{(1)}\right)$ is a polyhomogeneous two-sided ideal of $A\left(X^{(1)}\right)$. So, the variety Θ can be defined by the identities, which have a form $fx^{(2)}=0$, where f is a polyhomogeneous element of the ideal $S_\Theta\left(X^{(1)}\right)$. $f=\sum_{i\in I}\lambda_im_i$, where $|I|<\infty$, m_i are monomials of $A\left(X^{(1)}\right)$. By our assumption, we can suppose that $\lambda_i\in\mathbb{Z}$. So, as in [13, proof of the Theorem 4.1], $\widetilde{s}_F\left(fx^{(2)}\right)=\sum_{i\in I}\varphi\left(\lambda_i\right)a^{\deg f}m_ix^{(2)}=$

$a^{\deg f}\sum\limits_{i\in I}\lambda_i m_i x^{(2)} = a^{\deg f} f x^{(2)} \in S_\Theta\left(X^{(1)}\right) x^{(2)}$. Hence, there exists a homomorphism $s_{F_\Theta}: F_\Theta \to \left(F_\Theta\right)_W^*$, such that $s_{F_\Theta}\nu = \nu\widetilde{s}_F$. In particular, if $x \in X$, then $s_{F_\Theta}\nu(x) = \nu(x)$, or, in our short notation, $\left(s_{F_\Theta}\right)_{|X} = id_X$.

As in [12, proof of Theorem 5.4], we can prove, at first, that s_{F_Θ} is an isomorphism, and then, by Criterion 4.1, that the strongly stable automorphism which corresponds to the system of words (4.1) is inner if and only if $\varphi = id_k$. Therefore $\mathfrak{A}/\mathfrak{Y} \cong$Aut$k$. ■

5 Example

In this section we consider a field k such that Aut$k \neq \{id_k\}$. We will give an example of the variety Θ of representations, which fulfills the conditions of the Theorem 4.1, and two representations $H_1, H_2 \in \Theta$, such that they are automorphically equivalent, but not geometrically equivalent. This example is similar to the examples of [13, Example 3] and [12, Subsection 5.4].

We consider the variety Θ of representations, defined by identity

$$x_1^{(1)}x_2^{(1)}\ldots x_5^{(1)}x_6^{(1)}x^{(2)} = 0.$$

In this variety we consider the free algebra $F = F_\Theta\left(x_1^{(1)}, x_2^{(1)}, x^{(2)}\right)$. $A_\Theta\left(x_1^{(1)}, x_2^{(1)}\right)$ contains two linear independent elements

$$\iota\left[x_1^{(1)}, \left[x_1^{(1)}, \left[\left[x_1^{(1)}, x_2^{(1)}\right], x_2^{(1)}\right]\right]\right] = e_1$$

and

$$\iota\left[\left[x_1^{(1)}, \left[x_1^{(1)}, x_2^{(1)}\right]\right], \left[x_1^{(1)}, x_2^{(1)}\right]\right] = e_2.$$

We denote by W the system of words (4.1), such that $a = 1$, $\varphi \neq \{id_k\}$, and by Φ the strongly stable automorphism of the category Θ^0, defined by the system of words W. There exists $\lambda \in k$ such that $\varphi(\lambda) \neq \lambda$. We denote $t = \lambda e_1 + e_2$. We denote by T the two-sided ideal of the algebra $A_\Theta\left(x_1^{(1)}, x_2^{(1)}\right)$ generated by the element t. In truth, $T = \mathrm{sp}_k(t)$. We denote by H the representation such that $(H)^{(1)} = (F)^{(1)}$, $(H)^{(2)} = \left(A_\Theta\left(x_1^{(1)}, x_2^{(1)}\right)/T\right)x^{(2)}$. $H \in \Theta$ by the Birkhoff Theorem. $H_W^* \in \Theta$ by [12, Proposition 3.5]. By [12, Corollary 1 from the Proposition 4.2], the representations H and H_W^* are automorphically equivalent. By computations which are very similar to the computations of [13, Example 3] and [12, Subsection 5.4], H and H_W^* are not geometrically equivalent.

6 Acknowledgements

I would like to acknowledge the support of PNPD CAPES – Programa Nacional de Pós-Doutorado da Coordenação de Aperfeiçoamento de Pessoal de Nível Su-

perior (National Postdoctoral Program of the Coordination for the Improvement of Higher Education Personnel, Brazil) and of CNPq - Conselho Nacional de Desenvolvimento Científico e Tecnológico (National Council for Scientific and Technological Development, Brazil), Project 314045/2013-9, for bestowing a visiting researcher scholarship.

I also would like to acknowledge Professors Nir Cohen and David Armando Zavaleta Villanueva, who initiated my participation in these two programs.

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